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Representations of the quantum algebra $U_{q,s}(su_{1,1})$

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Abstract. All irreducible representations are given for the two-parameter quantum algebra $U_{q,s}(su_{1,1})$. They are determined by two complex numbers. Infinitesimally unitary representations of $U_{q,s}(su_{1,1})$ are defined (they are counterparts of infinitesimally unitary representations of the Lie algebra $su_{1,1}$) and separated from the set of irreducible representations. As in the case of the quantum algebra $U_q(su_{1,1})$, there are the principal unitary series, the supplementary series, the strange series and the discrete series of infinitesimally unitary representations of $U_{q,s}(su_{1,1})$. The strange series disappears when $q, s \to 1$. It is shown that for $s^{-1} \leqslant q \leqslant s$ the operators $T_a^+(E_+)$ and $T_a^+(E_-)$ of representations of the positive discrete series are bounded. For $s \leqslant q \leqslant s^{-1}$ the operators $T_a^-(E_+)$ and $T_a^-(E_-)$ of representations of the negative discrete series are bounded.

1. Introduction

Quantum groups and algebras are of great importance for applications in quantum integrable systems, in quantum field theory, and in statistical physics. To apply them it is necessary to have a well developed theory of their representations. Representations of the simplest quantum algebras are of great significance for applications. There is, more or less, clearness about the finite-dimensional representations of quantum groups and algebras: non-equivalent irreducible representations have been classified, uniqueness of highest weights has been proved, relations to irreducible representations of Lie groups and algebras have been shown, and so on (Rosso 1988). Infinite-dimensional representations of the 'non-compact' quantum algebra $U_q(su_{1,1})$ have also been investigated. It has been shown that closures of the symmetric operators $T(E_+ + E_-)$ and $T(\mathrm{i}E_+ - \mathrm{i}E_-)$ of infinitesimally unitary operators are not self-adjoint operators (Burban and Klimyk 1993). This situation is new with respect to representations of the Lie algebra su(1,1).

There are two-parameter quantum algebras and groups which are also important for applications in physics. The two-parametric quantum algebra $U_{q,s}(sl_2)$ was studied by Schirrmacher *et al* (1991), and Chakrabarti and Jagannathan (1991) constructed finite-dimensional representations of this algebra and connected it with the two-parameter deformed oscillator. Finite-dimensional representations of the quantum algebra $U_{q,s}(sl_2)$ were also considered by Smirnov and Wehrhahn (1992). These representations are in a one-to-one correspondence with finite-dimensional representations of the Lie algebra su(1,1). Jing (1993) also studied a connection between the quantum algebra $U_{q,s}(su_2)$ and the deformed two-parameter harmonic oscillator. The Jordan–Schwinger realization of $U_{q,s}(su_2)$ has been constructed, and the Clebsch–Gordan coefficients for representations of $U_{q,s}(su_2)$ have been evaluated (Smirnov and Wehrhahn 1992).

In this paper we begin to study representations of the 'non-compact' quantum algebra $U_{q,s}(su_{1,1})$. These are of great importance because of their connection with the deformed

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harmonic oscillator. We give a complete classification of irreducible representations of $U_{q,s}(su_{1,1})$. Then we define infinitesimally unitary representations of $U_{q,s}(su_{1,1})$ which are counterparts of infinitesimally unitary representations of the Lie algebra $su_{1,1}$. All infinitesimally unitary representations of $U_{q,s}(su_{1,1})$ are separated from the set of irreducible representations. Thus, we obtain a classification of infinitesimally unitary representations of $U_{q,s}(su_{1,1})$. These representations are in a one-to-one correspondence with infinitesimally unitary representations of the one-parameter quantum algebra $U_q(su_{1,1})$. However, infinite-dimensional representations of $U_{q,s}(su_{1,1})$ have some peculiarities. Namely, the operators $T_a^+(E_+)$ and $T_a^+(E_-)$ of representations of the positive discrete series are bounded if $s^{-1} \leqslant q \leqslant s$. The operators $T_a^-(E_+)$ and $T_a^-(E_-)$ are bounded if $s \leqslant q \leqslant s^{-1}$. We do not have this situation for the the quantum algebra $U_q(su_{1,1})$ or for the Lie algebra su(1,1).

In section 2 we define the quantum algebra $U_{q,s}(su_{1,1})$. In section 3 the standard representations $T_{a\epsilon}$ of $U_{q,s}(su_{1,1})$ are constructed. A classification of irreducible representations is given in section 4. In section 5 we separate infinitesimally unitary representations of $U_{q,s}(su_{1,1})$ from the set of irreducible representations.

2. The quantum algebra $U_{q,s}(su_{1,1})$

Let q and s be fixed complex numbers. The elements H, E_+, E_- satisfying the commutation relations

$$[H, E_+] = E_+ \qquad [H, E_-] = -E_- \tag{1}$$

$$[E_+, E_-]_s \equiv s^{-1/2} E_+ E_- - s^{1/2} E_- E_+ = s^{-H} \frac{q^H - q^{-H}}{q^{1/2} - q^{-1/2}} \tag{2}$$

generate an associative algebra (here $[H, E_\pm]$ is the usual commutator). This algebra (denoted by A) is called the two-parameter deformation of the universal enveloping algebra $U(sl_2)$ of the Lie algebra sl(2,ℂ). The structure of a Hopf algebra can be introduced into A (Schirrmacher *et al* 1991). The algebra A with this structure is called the two-parameter quantum algebra $U_{q,s}(sl_2)$.

Sometimes, instead of q and s, the parameters

$$q' = s^{-1} q \qquad p = sq$$

are used. In this case relation (2) is replaced by

$$[E_+, E_-]_{pq} \equiv E_+ E_- - (pq^{-1})^{1/2} E_- E_+ = \frac{q^H - p^{-H}}{q^{1/2} - p^{-1/2}}$$

where we write q instead of q'.

If q and s are not roots of unity, then the centre of $U_{q,s}(sl_2)$ is generated by one Casimir element

$$C = s^H (s E_- E_+ + [H]_{q,s} [H+1]_{q,s}) \tag{3}$$

where

$$[a]_{q,s} = \frac{(s^{-1}q)^{a/2} - (sq)^{-a/2}}{(s^{-1}q)^{1/2} - (sq)^{-1/2}} = s^{(1-a)/2} [a]$$
$$[a] = \frac{q^{a/2} - q^{-a/2}}{q^{1/2} - q^{-1/2}}. \tag{4}$$

This element commutes with all elements of $U_{q,s}(sl_2)$.

One can introduce *-structures into the Hopf algebra $U_{q,s}(sl_2)$ which turn this algebra into *-Hopf algebras. These are analogues of real forms of the complex Lie algebra sl(2,$\mathbb{C}$). If $q, s \in \mathbb{R}$, then the *-structure generated by the relations

$$H^* = H \qquad E_+^* = -E_- \qquad E_-^* = -E_+ \tag{5}$$

gives the quantum algebra $U_{q,s}(su_{1,1})$, which is an analogue of the real form su(1,1) of the Lie algebra sl(2,$\mathbb{C}$).

By a linear representation T of the algebra $U_{q,s}(sl_2)$ we mean a homomorphism of $U_{q,s}(sl_2)$ into the algebra of linear operators (bounded or unbounded) on a Hilbert space, defined on a uniformly dense invariant subspace D, such that the operator $T(H)$ can be diagonalized and has a discrete spectrum. Such representations of $U_{q,s}(sl_2)$ lead to linear representations of the associative algebra $U_{q,s}(su_{1,1})$ which in general are not representations of the *-Hopf algebra $U_{q,s}(su_{1,1})$, since relations (5) can be violated for the operators $T(H)$, $T(E_+)$ and $T(E_-)$.

To determine a representation T of $U_{q,s}(sl_2)$ it is sufficient to give the operators $T(E_+)$, $T(E_-)$, $T(H)$ for which relations (1) and (2) are fulfilled on a uniformly dense subspace D. If in addition the equalities

$$T(H)^* = T(H) \qquad T(E_+)^* = -T(E_-) \tag{6}$$

are satisfied on D, then T is called an infinitesimally unitary representation of the associative algebra $U_{q,s}(su_{1,1})$. In this case T is a representation of the quantum algebra (of the *-Hopf algebra) $U_{q,s}(su_{1,1})$ which is also called a *-representation.

If $s = 1$ then $U_{q,s}(su_{1,1})$ coincides with the one-parameter quantum algebra $U_q(su_{1,1})$. In the papers by Klimyk and Groza (1989), Masuda *et al* (1990) and Vaksman and Korogodsky (1990) representations T (unitary and non-unitary) of $U_q(su_{1,1})$ are considered, for which the spectrum of the operator $T(H)$ consists of integers or half-integers. Infinite-dimensional representations of $U_q(su_{1,1})$ parametrized by two complex numbers are studied by Burban and Klimyk (1993). Spectra of the operators $T(H)$ for these representations can consist of complex numbers. These representations are a q-analogue of representations of the universal covering group for the Lie group SU(1,1). However, for representations of $U_q(su_{1,1})$ there are some peculiarities which are absent in the classical case. In this paper we deal with representations of $U_{q,s}(su_{1,1})$. When $s = 1$ we obtain representations of $U_q(su_{1,1})$ from the paper by Burban and Klimyk (1993).

3. Infinite-dimensional representations

Let q and s be positive real numbers. We construct infinite-dimensional representations of $U_{q,s}(su_{1,1})$ which are an analogue of the representations $T_{a\epsilon}$ in the paper by Burban and Klimyk (1993). Let ϵ be a fixed complex number, which defines the complex Hilbert space V_ϵ with the orthonormal basis

$$\{|m\rangle;\ m = n + \epsilon,\ n = 0, \pm 1, \pm 2, \ldots\}. \tag{7}$$

For every complex number a we construct the representation $T_{a\epsilon}$ of the associative algebra $U_{q,s}(sl_2)$ (and of the associative algebra $U_{q,s}(su_{1,1})$) on V_ϵ defined by

$$T_{a\epsilon}(H)|m\rangle = m|m\rangle \tag{8}$$

$$T_{a\epsilon}(E_+)|m\rangle = -s^{(a-m)/2}[a-m]|m+1\rangle \tag{9}$$

$$T_{a\epsilon}(E_-)|m\rangle = -s^{-(a+m)/2}[a+m]|m-1\rangle \tag{10}$$

where $[b]$ is given by (4). A direct verification shows that the operators (8)–(10) are defined on the everywhere dense subspace D of the Hilbert space V_ϵ consisting of finite linear combinations of the basis elements (7) and transform D into D. Moreover, they satisfy relations (1) and (2). By making use of (3) we easily find that

$$T_{a\epsilon}(C)|m\rangle = s^{1/2}[a][a+1]|m\rangle. \tag{11}$$

Replacing the basis (7) by the basis

$$\{|m\rangle' = s^{am/2}|m\rangle,\ m = n+\epsilon,\ n = 0, \pm 1, \pm 2, \ldots\}$$

we obtain

$$T_{a\epsilon}(H)|m\rangle' = m|m\rangle' \tag{12}$$

$$T_{a\epsilon}(E_+)|m\rangle' = -s^{-m/2}[a-m]|m+1\rangle' \tag{13}$$

$$T_{a\epsilon}(E_-)|m\rangle' = -s^{-m/2}[a+m]|m-1\rangle'. \tag{14}$$

These equations show that the representations $T_{a\epsilon}$ and $T_{a,\epsilon+k}$ coincide for $k \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers. Therefore, we restrict ourselves to the cases where

$$0 \leqslant \operatorname{Re}\epsilon < 1.$$

Because of the periodicity of the function $w(z) = [z]$ (Burban and Klimyk 1993) we have the equivalence relations

$$T_{a\epsilon} \sim T_{a+4\pi k\mathrm{i}/h,\epsilon} \qquad k \in \mathbb{Z}$$

where h is defined by the relation $q = \exp h$. Since $w(z) = -w(z + 2\pi\mathrm{i}/h)$, then the replacement of a by $a + 2\pi\mathrm{i}/h$ only changes the signs on the left-hand sides of (13) and (14). This means that

$$T_{a\epsilon} \sim T_{a+2\pi k\mathrm{i}/h,\epsilon} \qquad k \in \mathbb{Z}.$$

4. Irreducible representations

Irreducibility of the representations $T_{a\epsilon}$ is analysed in the same way as in the case of the quantum algebra $\mathrm{U}_q(\mathrm{su}_{1,1})$ (Burban and Klimyk 1993). This analysis leads to the following theorem:

Theorem 1. The representation $T_{a\epsilon}$ of the algebra $\mathrm{U}_{q,s}(\mathrm{sl}_2)$ (and of the algebra $\mathrm{U}_{q,s}(\mathrm{su}_{1,1})$) is irreducible if and only if $a \not\equiv \pm\epsilon \pmod 1$. If $\epsilon = 0$ or $\epsilon = \frac{1}{2}$ then these inequalities are replaced by one condition $a \not\equiv \epsilon \pmod 1$.

There exist the equivalence relations $T_{a\epsilon} \sim T_{-a-1,\epsilon}$ in the set of irreducible representations $T_{a\epsilon}$. The equivalence operator is evaluated in the same way as in the classical case (Vilenkin and Klimyk 1991, section 6.4.4). For $T_{a\epsilon}$ and $T_{-a-1,\epsilon}$ the equivalence operator A is diagonal with respect to the basis $\{|m\rangle'\}$ and its diagonal elements are of the form

$$d_m = \frac{[a+\epsilon+1][a+\epsilon+2]\cdots[a+m]}{[-a+\epsilon][-a+\epsilon+1]\cdots[-a+m-1]}.$$

It is possible to show that we have described all equivalence relations for the irreducible representations $T_{a\epsilon}$ of $U_{q,s}(su_{1,1})$.

Let us consider reducible representations $T_{a\epsilon}$. In the carrier space of the representation $T_{a\epsilon}$ invariant subspaces appear because of the vanishing of some of the coefficients

$$[-a+m] = [-a+\epsilon+n] \qquad [-a-m] = [-a-\epsilon-n] \qquad n \in \mathbb{Z}$$

from (13) and (14). Conditions of reducibility of the representations $T_{a\epsilon}$ of the quantum algebra $U_q(su_{1,1})$ are exactly the same (Burban and Klimyk 1993). Therefore, in order to study the reducible representations $T_{a\epsilon}$ of $U_{q,s}(su_{1,1})$ we have to repeat word for word the corresponding consideration in the paper by Burban and Klimyk (1993). As a result, we obtain classes of irreducible representations of $U_{q,s}(su_{1,1})$ which are subrepresentations of reducible representations $T_{a\epsilon}$. Let us describe these irreducible representations.

If $\epsilon = 0$ or $\epsilon = \frac{1}{2}$ then we have the irreducible representations T_l^+, $l = -\frac{1}{2}, 0, \frac{1}{2}, 1, \ldots$, and T_l^-, $l = \frac{1}{2}, 0, -\frac{1}{2}, -1, \ldots$. The spectrum of the operator $T_l^+(H)$ coincides with $l+1, l+2, l+3, \ldots$ and that for the operator $T_l^-(H)$ is $l-1, l-2, l-3, \ldots$. If $\epsilon \neq 0$ and $\epsilon \neq \frac{1}{2}$ then we have pairwise non-equivalent irreducible representations T_a^- and T_a^+, where a is a complex number such that $a \not\equiv 0 \pmod 1$ and $a \not\equiv \frac{1}{2} \pmod 1$. The spectrum of the operator $T_a^-(H)$ coincides with the set of points $a-1-k$, $k = 0, 1, 2, \ldots$, and that of the operator $T_a^+(H)$ is $a+1+k$, $k = 0, 1, 2, \ldots$. The representations T_l^+ and T_a^+ are determined by (12)–(14), in which a is replaced respectively by $-l-1$ and by $-a-1$. The representations T_l^- and T_a^- are given by (12)–(14) in which a is replaced respectively by $l-1$ and by $a-1$.

Thus, if $\epsilon = 0$ or $\epsilon = \frac{1}{2}$ then we have the following classes of irreducible representations of $U_{q,s}(su_{1,1})$:

(a) the representations $T_{a\epsilon}$ where $\operatorname{Re} a \geqslant -\frac{1}{2}$, $a \not\equiv \epsilon \pmod 1$ and $0 \leqslant \operatorname{Im} a < 2\pi/h$;
(b) the representations T_l^+ and T_{-l}^-, $l = -\frac{1}{2}, 0, \frac{1}{2}, 1, \ldots$;
(c) finite-dimensional irreducible representations.

If $\epsilon \not\equiv 0 \pmod 1$ and $\epsilon \not\equiv \frac{1}{2} \pmod 1$ (recall that $0 \leqslant \operatorname{Re}\epsilon < 1$), then we have the following classes of irreducible representations:

(a) the representations $T_{a\epsilon}$, $a \not\equiv \pm\epsilon \pmod 1$, $\operatorname{Re} a \geqslant -\frac{1}{2}$ and $0 \leqslant \operatorname{Im} a < 2\pi/h$;
(b) the representations T_a^+, T_a^-, $a \not\equiv 0 \pmod 1$, $a \not\equiv \frac{1}{2} \pmod 1$.

The associative algebra $U_{q,s}(su_{1,1})$ has no other algebraically irreducible representations. This assertion is proved with the help of (3) and (11) in the same manner as in the case of the algebra $U_q(su_{1,1})$ (Burban and Klimyk 1993).

Let us consider the problem of boundedness of the operators $T(H)$, $T(E_+)$ and $T(E_-)$ of irreducible representations. It is clear from (12) that the operator $T(H)$ is unbounded for all infinite-dimensional irreducible representations. The coefficients in (13) and (14) are not bounded when m take the values $\epsilon+n$, $n = 0, \pm 1, \pm 2, \ldots$. Therefore, the operators $T_{a\epsilon}(E_+)$ and $T_{a\epsilon}(E_-)$ are unbounded for any a and ϵ. This is not the case for the representations T_a^+ and T_a^-. The coefficients from (13) and (14) can be written down as

$$-s^{-m/2}[a-m] = -\{q^{a/2}(qs)^{-m/2} - q^{-a/2}(q/s)^{m/2}\}(q^{1/2} - q^{-1/2})^{-1}$$

$$-s^{-m/2}[a+m] = -\{q^{a/2}(q/s)^{m/2} - q^{-a/2}(qs)^{-m/2}\}(q^{1/2} - q^{-1/2})^{-1}.$$

Values of m in the representations T_a^+ are bounded below. Therefore, these coefficients are bounded if and only if $qs \geqslant 1$ and $q/s \leqslant 1$. This means that the operators $T_a^+(E_+)$ and $T_a^+(E_-)$ are bounded if and only if

$$s^{-1} \leqslant q \leqslant s.$$

In this case $s > 1$ (we assume that $q \neq 1$). In the same manner we derive that the operators $T_a^-(E_+)$ and $T_a^-(E_-)$ are bounded if and only if

$$s \leqslant q \leqslant s^{-1}.$$

In this case $s < 1$. We do not have this situation for representations of the Lie algebra su(1,1). Moreover, the operators $T(E_+)$ and $T(E_-)$ are unbounded for all infinite-dimensional irreducible representations of the quantum group $\mathrm{U}_q(\mathrm{su}_{1,1})$, $q \in \mathbb{R}$ (Burban and Klimyk 1993). Notice that $T(E_+)$ and $T(E_-)$ are bounded for all irreducible representations of the associative algebra $\mathrm{U}_q(\mathrm{su}_{1,1})$ when q is not real and lies on the unit circle.

5. Infinitesimally unitary representations

Let D be the linear subspace in the carrier Hilbert space V_ϵ of the representation $T_{a\epsilon}$ spanned by the basis vectors $|m\rangle$, $m = \epsilon + n$, $n \in \mathbb{Z}$. Let us find for which representations $T_{a\epsilon}$ relations (6) are fulfilled on D. It is clear that the condition $T(H)^* = T(H)$ means that the spectrum of the operator $T(H)$ is real; that is, $0 \leqslant \epsilon < 1$. The condition $T(E_+)^* = -T(E_-)$ means that for all $m = \epsilon + n$, $n \in \mathbb{Z}$, the condition

$$-a + m - 1 = \overline{a + m}$$

must be satisfied, where the bar denotes complex conjugation. This condition is fulfilled if and only if $a = \mathrm{i}\rho - \frac{1}{2}$, $\rho \in \mathbb{R}$. Thus, the representations $T_{\mathrm{i}\rho-1/2,\epsilon}$, $0 \leqslant \epsilon < 1$, are unitary. They form the principal unitary series of representations of the quantum algebra $\mathrm{U}_{q,s}(\mathrm{su}_{1,1})$.

In order to find other infinitesimally unitary representations of $\mathrm{U}_{q,s}(\mathrm{su}_{1,1})$ we introduce a new basis in the space V_ϵ. Assuming that a and ϵ are real numbers, we set $|m\rangle'' = s^m|m\rangle$. Then

$$T_{a\epsilon}(H)|m\rangle'' = m|m\rangle''$$

$$T_{a\epsilon}(E_+)|m\rangle'' = -[a-m]_{q,s^{-1}}|m+1\rangle''$$

$$T_{a\epsilon}(E_-)|m\rangle'' = -[a+m]_{q,s}|m-1\rangle''$$

where $[b]_{q,s} = s^{(1-b)/2}[b]$. Now we put

$$|m\rangle'' = \left(\frac{[a+\epsilon+r+1]_{q,s}[a+\epsilon+r+2]_{q,s}\cdots[a+r+m]_{q,s}(-1)^m}{[a-\epsilon-r]_{q,s^{-1}}[a-\epsilon-r-1]_{q,s^{-1}}\cdots[a-r-m+1]_{q,s^{-1}}}\right)^{1/2}|m\rangle'''$$

where r is an integer. The operators of the representation $T_{a\epsilon}$ are given in the basis $\{|m\rangle'''\}$ by

$$T_{a\epsilon}(H)|m\rangle''' = m|m\rangle''' \tag{15}$$

$$T_{a\epsilon}(E_+)|m\rangle''' = \mathrm{i}([a-m]_{q,s^{-1}}[a+m+1]_{q,s})^{1/2}|m+1\rangle''' \tag{16}$$

$$T_{a\epsilon}(E_-)|m\rangle''' = \mathrm{i}([a-m+1]_{q,s^{-1}}[a+m]_{q,s})^{1/2}|m-1\rangle'''. \tag{17}$$

The operators of the representations T_l^+, T_l^-, T_a^+, T_a^- are given in the basis $\{|m\rangle'''\}$ by the same equations with the replacement indicated above. For these representations, values of the parameter m are bounded from below or from above.

We now verify for which irreducible representations of $U_{q,s}(su_{1,1})$ relations (6) are satisfied for operators (15)–(17). This verification is done in the same manner as in the classical case. Such analysis gives us the following additional classes of unitary irreducible representations of $U_{q,s}(su_{1,1})$:

(a) the representations $T_{a\epsilon}$ with $0 \leqslant \epsilon < 1$, where $\epsilon - 1 > a > -\epsilon$ for $\epsilon > \frac{1}{2}$ and $-\epsilon > a > \epsilon - 1$ for $\epsilon < \frac{1}{2}$ (the supplementary series);
(b) the representations $T_{a\epsilon}$, $\operatorname{Im} a = \pi/h$, $\operatorname{Re} a > -\frac{1}{2}$ (the strange series);
(c) all representations T_a^+, $a \geqslant -\frac{1}{2}$ and T_a^-, $a \leqslant \frac{1}{2}$ (the discrete series);
(d) the zero representation.

It can be shown, by using the method of section 6.4.6 of the book by Vilenkin and Klimyk (1991), that the irreducible representations, listed here, exhaust all irreducible infinitesimally unitary representations of $U_{q,s}(su_{1,1})$.

6. Conclusion

We have found all irreducible representations of the algebra $U_{q,s}(su_{1,1})$. They are given by two complex parameters. We defined infinitesimally unitary representations of $U_{q,s}(su_{1,1})$ (they are counterparts of infinitesimally unitary representations of the Lie algebra $su_{1,1}$). Then we separated all infinitesimally unitary representations in the set of irreducible representations of $U_{q,s}(su_{1,1})$. There are the principal unitary series, the supplementary series, the strange series and the discrete series of infinitesimally unitary representations. The strange series disappears when $q, s \to 1$.

Irreducible representations of $U_q(su_{1,1})$ were constructed with the help of the so-called standard representations $T_{a\epsilon}$ of this algebra. In the set of representations $T_{a\epsilon}$ there exist equivalence relations which are consequences of the periodicity of the function $w(z) = [z]$. Other equivalence relations are related to existence of intertwining operators for the representations $T_{a\epsilon}$ and $T_{-a-1,\epsilon}$. These intertwining operators are similar to those for representations of the group SU(1,1) (Vilenkin and Klimyk 1991).

Comparing irreducible representations of the quantum algebra $U_{q,s}(su_{1,1})$ with those of the algebra $U_q(su_{1,1})$ we conclude that there exists a one-to-one correspondence between irreducible representations of these algebras. We also have a one-to-one correspondence between unitary irreducible representations of $U_{q,s}(su_{1,1})$ and those of $U_q(su_{1,1})$. Nevertheless, representations of $U_{q,s}(su_{1,1})$ have some peculiarities. Namely, the operators $T_a^+(E_+)$ and $T_a^+(E_-)$ of representations of the discrete series of $U_{q,s}(su_{1,1})$ are bounded if $s^{-1} \leqslant q \leqslant s$. The operators $T_a^-(E_+)$ and $T_a^-(E_-)$ are bounded if $s \leqslant q \leqslant s^{-1}$. In the first case $s > 1$ and in the second case $s < 1$.

It is known (Burban and Klimyk 1993) that closures of the symmetric operators $T(E_+ + E_-)$ and $T(\mathrm{i}E_+ - \mathrm{i}E_-)$ of infinitesimally unitary representations of the quantum group $\mathrm{U}_q(\mathrm{su}_{1,1})$ are not self-adjoint, but they have self-adjoint extensions. It is interesting to investigate these operators for infinitesimally unitary representations of the algebra $\mathrm{U}_{q,s}(\mathrm{su}_{1,1})$.

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